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On Cardinal Perfect Splines of Least Sup–Norm on the Real Axis

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

During the recent past, there has been considerable interest in various classes of polynomial spline functions and the minimization problems which arise when these classes are endowed with the Chebyshev norm. The prototype of all these problems is Chebyshev's: Find the polynomial $P_{m-1}(x)$ of degree at most $m - 1$ such that the quantity

$$\max_{-1 \leq x \leq 1} |x^m - P_{m-1}(x)|$$

is minimal; this leads to the classical monic Chebyshev polynomials. Much later the problem was generalized to the case of monosplines (see [3, 9]). For example, Johnson considered the problem of minimizing the quantity

$$\max_{-1 \leq x \leq 1} |x^m - S_{m-1,k}(x)|,$$

where $S_{m-1,k}(x)$ is any spline function of degree $m - 1$ with k simple knots, and he was able to characterize the unique solution to this problem in terms of certain alternation properties.

More recently, Schoenberg and the present author have been considering another important class of functions: the perfect splines. To define this class let m and k be natural numbers. A function $P(x)$ is a *perfect spline* of degree m having k distinct simple knots in the open interval $(-1, 1)$ if there are points $-1 < x_1 < x_2 < \cdots < x_k < 1$ such that

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(a) the restrictions of $P(x)$ to each of the intervals $(-1, x_1)$, $(x_1, x_2), \ldots, (x_{k-1}, x_k)$, $(x_k, 1)$ are polynomials of degree m,

(b) $P(x) \in C^{m-1}[-1, 1]$,

(c) $|P^{(m)}(x)| = m!$, except at the knots, where the mth derivative may fail to exist.

We denote the class of such functions by $\mathscr{P}_{m,k}$. This definition is due to Glaeser [2]. However, earlier such functions played a central role in some of Favard's work on interpolation with functions whose mth derivatives are of minimal sup-norm [1].

While deriving best possible inequalities between the norms of the derivatives of a function defined on the half line [8], Schoenberg and the author were lead once again to the notion of a perfect spline. As a small part of this work, we considered the following problem of Chebyshev type: Determine within the class of perfect splines $\mathscr{P}_{m,k}$ the perfect spline $P_{m,k}(x)$ of least L_∞-norm on $[-1, 1]$, i.e., find $P_{m,k}$ such that

$$\|P_{m,k}\|_\infty = \inf_{P \in \mathscr{P}_{m,k}} \sup_{-1 \leq x \leq 1} |P(x)|. \tag{1.1}$$

In the case where $k = 0$, this reduces to the classical Chebyshev problem. Since the k knots are themselves variable, we see that the family $\mathscr{P}_{m,k}$ depends on $m + k$ parameters, and so we should expect the optimal solution to have $m + k + 1$ points at which the extreme values $\pm \|P_{m,k}\|_\infty$ are assumed with alternating sign. This indeed turned out to be the case and we established the following theorem [8].

THEOREM. *There is a unique $P_{m,k}(x)$ satisfying* (1.1) *and it is a perfect spline of degree m with k simple knots. Moreover, $P_{m,k}(x)$ has precisely $m + k + 1$ points of equioscillation, and this characterizes the optimal solution to* (1.1).

As a simple illustration of this theorem, we can construct explicitly $P_{2,k}$ for arbitrary k, say $k = 3$. Let $T_2(x)$ be the quadratic Chebyshev polynomial pictured in Fig. 1, and let $\alpha < \beta$ be its two zeros. In Fig. 2, we have started with $T_2(x)$, cut off at β. Then we have attached on arcs of $T_2(x)$ restricted to $[\alpha, \beta]$, but inverting them as necessary to obtain a C^1 composite function. Finally on $[\xi_3, b]$ we describe an arc identical to $-T_2(x)$ restricted to $[\alpha, 1]$.

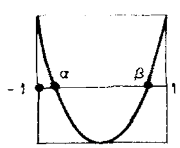

FIGURE 1

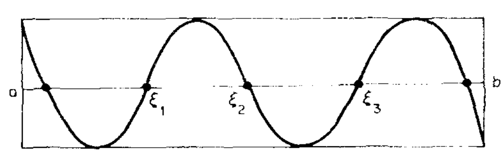

FIGURE 2

Thus, $\xi_2 - \xi_1 = \xi_3 - \xi_2 = \beta - \alpha$. By a change of variables converting the interval $[a, b]$ to $[-1, 1]$ and normalizing the new function so that its highest degree coefficients are ± 1, we obtain a perfect spline satisfying the conditions of the above theorem. Indeed, $m + k + 1 = 2 + 3 + 1 = 6$, which is precisely the number of extrema exhibited in Fig. 2.

For higher degrees, constructions analogous to the above break down because we fail to get composite functions of a sufficiently high continuity class. However, by using the methods of linear programming together with differential corrections for the nonlinear parameters, these optimal perfect splines can be computed with some accuracy (see [8]).

One of the beauties of spline function theory is that, unlike polynomials, a spline function can remain bounded on the whole real axis, provided, of course, it can have infinitely many knots. This being the case, it is natural to pose certain Chebyshev-like problems for splines on the entire real axis, instead of a finite interval. For the monospline case, the problem was elegantly handled by Schoenberg and Ziegler [9], and we wish now to investigate the perfect spline case. Specifically we consider the following class of functions.

DEFINITION. Let m by any natural number and let $r = -1, 0, 1, ..., m - 1$. The class $\mathscr{P}_m^{\,r} = \{P(x)\}$ consists of all functions with the following two properties:

(i) $P(x) \in C^r(R)$.

(ii) Let ν be any integer. $P(x)$ restricted to $[2\nu, 2\nu + 1]$ is a polynomial of degree m with highest term x^m. $P(x)$ restricted to $[2\nu - 1, 2\nu]$ is a polynomial of degree m with highest term $-x^m$.

The case when $r = -1$ means that (i) is vacuous, there being no continuity requirements between the separate polynomial components of $P(x)$. Clearly $|P^{(m)}(x)| = m!$, except at the integers where the mth derivative is undefined, and so the word "perfect," in the sense of Glaeser, applies to our class. Also, the knots have been fixed at the integers, and so we deal with the so-called cardinal splines [7]. Therefore, it is appropriate to call the functions defined above cardinal perfect splines of degree m with $m - r$ fold knots. Our main concern then is the following problem.

Problem. To determine

$$P(x) \in \mathscr{P}_m^{\,r}$$

having least Chebyshev norm

$$\|P\|_\infty = \sup_{x \in R} |P(x)|.$$

This problem can be solved using the ideas of Schoenberg and Ziegler [9]. Our results complement and in some sense complete their work.

2. The Cases $r = -1$ and $r = m - 1$

We can dispose of these two extreme cases rather easily. The problem is indeed quite trivial when $r = -1$, for then there is no continuity requirement at the integers. To describe the optimal perfect spline, let us denote by $T_m(x)$ the usual Chebyshev polynomial for $[-1, 1]$. Set

$$P(x) = \begin{cases} 2^{-2m+1}T_m(2x-1) & 0 \leq x \leq 1, \\ -2^{-2m+1}T_m(2x-3) & 1 \leq x \leq 2, \end{cases}$$

and extend this definition periodically with period 2. So $P(x) \in \mathscr{S}_m^{-1}$ and clearly solves the problem for this case.

For $r = m - 1$ we need the so-called Euler polynomials $E_m(x)$, defined as the polynomial solution of the functional equation

$$\tfrac{1}{2}(f(x+1) + f(x)) = x^m. \tag{2.1}$$

It is easily seen that the Euler polynomials satisfy the boundary conditions

$$E_m^{(\nu)}(0) = -E_m^{(\nu)}(1) \qquad \nu = 0, 1,..., m-1 \tag{2.2}$$

if $m \geq 1$ and that they are thereby determined up to an additive constant and a multiplicative scalar factor. From these relations we see that the extension $\bar{E}_m(x)$, defined by

$$\bar{E}_m(x) = E_m(x) \qquad \text{for} \quad 0 \leq x < 1 \tag{2.3}$$

and

$$\bar{E}_m(x+1) = -\bar{E}_m(x) \qquad \text{for all } x, \tag{2.4}$$

is a spline function of period 2 with simple knots at the integers. Following Schoenberg [7] we call this composite function the Euler spline. Now from (2.1) we find that the Euler polynomials satisfy the differential equation

$$dE_m(x)/dx = mE_{m-1}(x), \tag{2.5}$$

and so

$$E_m^{(m)}(x) = m!E_0(x) = m!$$

whence

$$\bar{E}_m(x) \in \mathscr{S}_m^{m-1}. \tag{2.6}$$

We can now prove our first theorem.

Theorem 1. *Of all $P(x) \in \mathscr{S}_m^{m-1}$, only $\bar{E}_m(x)$ has least sup-norm.*

Proof. For suppose to the contrary that there exists $P(x) \in \mathscr{P}_m^{m-1}$ and $\|P(x)\|_\infty < \|\bar{E}_m(x)\|_\infty$. Then $S(x) = \bar{E}_m(x) - P(x)$ is a polynomial of degree at most $m - 1$. Because of the oscillatory behavior of $\bar{E}_m(x)$, $S(x)$ must have infinitely many zeros; hence, $S(x) \equiv 0$. ▌

3. Results for $0 \leqslant r \leqslant m - 2$

In this section we shall simply describe the results and defer the proofs to the later sections.

We have just seen that in the two extreme cases, our problem is solved by appropriate extensions of either the Chebyshev polynomials or the Euler polynomials. For the intermediate cases $0 \leqslant r \leqslant m - 2$, a solution will depend on a new type of polynomial, and since these new polynomials will enjoy a blend of the properties of the Chebyshev and Euler polynomials, we would like to call them the Euler–Chebyshev polynomials, or more shortly, ET-polynomials.

The construction of the ET-polynomials depends on a new property of the Euler polynomials, and so it will be helpful to recall what these classical polynomials look like; see Nörlund [5].

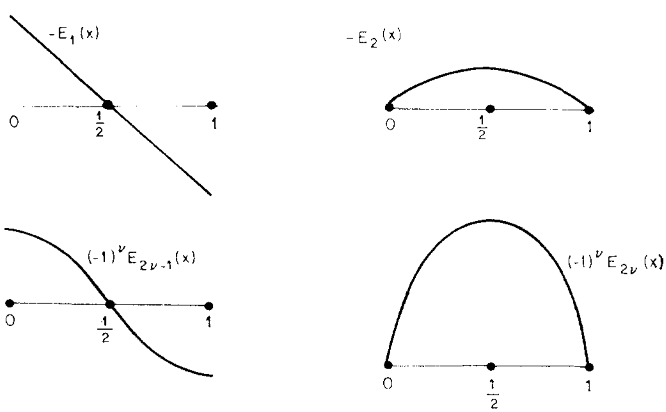

FIGURE 3

We note that the odd order Euler polynomials are odd about $x = \frac{1}{2}$ with a zero at $x = \frac{1}{2}$, while the even ones are even about $x = \frac{1}{2}$ and have zeros at $x = 0$ and $x = 1$. So the functions defined by

$$\tilde{E}_{2\nu-1}(x) = E_{2\nu-1}(x)/(x - \tfrac{1}{2}) \tag{3.1}$$

and

$$\tilde{E}_{2p}(x) = E_{2p}(x)/x \tag{3.2}$$

are polynomials.

Now let us recall the definition of a real Chebyshev set on an interval I. A set Φ of continuous real functions $\phi_1, \ldots, \phi_n$ defined on I is a *Chebyshev system* if the following condition is satisfied: Each nontrivial polynomial $P = a_1\phi_1 + \cdots + a_n\phi_n$ has at most $n - 1$ distinct zeros on I. Such systems are particularly important in approximation theory because of the characterization theorems for best approximation from such systems.

At any rate, with these definitions in mind we can state the following theorem.

THEOREM 2. *Let $1 \le p \le q$. Then the set of functions*

$$\tilde{E}_{2p}(x), \tilde{E}_{2p+2}(x), \ldots, \tilde{E}_{2q}(x)$$

forms a Chebyshev system on the closed interval $[0, \tfrac{1}{2}]$.

And we can also state the next theorem.

THEOREM 3. *Let $0 \le p \le q$. Then the set of functions*

$$\tilde{E}_{2p+1}(x), \tilde{E}_{2p+3}(x), \ldots, \tilde{E}_{2q+1}(x)$$

forms a Chebyshev system on the closed interval $[0, \tfrac{1}{2}]$.

We can also establish the following proposition, which is very similar to Chebyshev's theorem characterizing best approximations.

PROPOSITION 1. *Let $\{f_1(x), \ldots, f_k(x)\}$ be a Chebyshev system on $[a, b]$ and define*

$$g_i(x) = (x - a) f_i(x) \qquad i = 1, 2, \ldots, k. \tag{3.3}$$

Let $F(x)$ be a continuous function on $[a, b]$ vanishing at a. Then there exists a unique linear combination $\sum_{i=1}^{k} a_i g_i(x)$ of best approximation in the uniform norm to $F(x)$. This best approximation is uniquely characterized by a $k + 1$ point equioscillation property, i.e., there exist $k + 1$ points $a < x_1 < x_2 < \cdots < x_{k+1} \le b$ where the error function assumes the value of its norm with alternating signs.

We can now easily describe the ET-polynomials. Using Theorem 2 and Proposition 1 together with the fact that $E_{2q}(x)$ vanishes at $x = 0$, we define the ET-polynomial $E_{2q,2p-1}(x)$ as the unique polynomial of the form

$$E_{2q,2p-1}(x) = E_{2q}(x) + c_{q-p}E_{2q-2}(x) + \cdots + c_1 E_{2p}(x) \tag{3.4}$$

with the property that in the interval $[0, \frac{1}{2}]$

$$\| E_{2q,2p-1} \|_{\infty} = \text{minimum}. \tag{3.5}$$

COROLLARY 1. *The polynomial* $E_{2q,2p-1}(x)$ *is uniquely defined among all polynomials of the form* (3.4) *by the following equioscillation: There are* $q-p+1$ *points* x_ν *satisfying* $0 \leq x_1 < x_2 < \cdots < x_{q-p+1} \leq \frac{1}{2}$ *such that* $E_{2q,2p-1}(x_\nu)$ *assumes the value of* $\| E_{2q,2p-1} \|_\infty$ *with alternating signs as* ν *runs from* 1 *to* $q-p+1$.

In a similar way and using Theorem 3 together with the fact that $E_{2q+1}(\frac{1}{2}) = 0$, we define the ET-polynomial $E_{2q+1,2p}(x)$ as the unique polynomial of the form

$$E_{2q+1,2p}(x) = E_{2q+1}(x) + c_{q-p}E_{2q-1}(x) + \cdots + c_1 E_{2p-1}(x) \tag{3.6}$$

with the property that in the interval $[0, \frac{1}{2}]$

$$\| E_{2q+1,2p} \|_\infty = \text{minimum}. \tag{3.7}$$

COROLLARY 2. *The polynomial* $E_{2q+1,2p}(x)$ *is uniquely defined among all polynomials of the form* (3.6) *by the following equioscillation property: There are points* x_ν' *satisfying* $0 \leq x_1' < x_2' < \cdots < x'_{q-p+1} < \frac{1}{2}$ *such that* $E_{2q+1,2p}(x_\nu')$ *assumes the values* $\| E_{2q+1,2p} \|_\infty$ *with alternating signs as* ν *runs from* 1 *to* $q-p+1$.

Now using the ET-polynomials we can construct the desired optimal perfect splines in the cases where the degree and the order of the continuity class are of different parity.

Consider first $\mathscr{P}_{q2}^{2p-1}$. By (3.4) $E_{2q,2p-1}(x)$ is a linear combination of even order Euler polynomials and so is even about $x = \frac{1}{2}$. Furthermore, from the boundary conditions (2.2) enjoyed by the Euler polynomials, $E_{2q,2p-1}$ inherits the following boundary conditions:

$$E^{(\nu)}_{2q,2p-1}(0) = -E^{(\nu)}_{2q,2p-1}(1) \qquad \nu = 0, 1, \ldots, 2p-1. \tag{3.8}$$

So defining

$$\bar{E}_{2q,2p-1}(x) = E_{2q,2p-1}(x) \qquad \text{for} \quad 0 \leq x < 1 \tag{3.9}$$

and

$$\bar{E}_{2q,2p-1}(x+1) = -\bar{E}_{2q,2p-1}(x) \qquad \text{for all } x, \tag{3.10}$$

we conclude easily that

$$\bar{E}_{2q,2p-1}(x) \in \mathscr{P}_{2q}^{2p-1}.$$

We can now state the following theorem.

THEOREM 4. *Of all* $P(x) \in \mathscr{P}^{2q-1}_{2p}$, *only* $\bar{E}_{2q,2p-1}(x)$ *has least sup-norm.*

In a similar way we can handle $\mathscr{P}^{2q}_{2p-1}$. $E_{2q+1,2p}(x)$ is odd about $x = \frac{1}{2}$, and by (2.2) and (3.6) has the following boundary conditions:

$$E^{(\nu)}_{2q+1,2p}(0) = -E^{(\nu)}_{2q+1,2p}(1) \qquad \nu = 0, 1,\ldots, 2p.$$

So defining

$$\bar{E}_{2q+1,2p}(x) = E_{2q+1,2p}(x) \qquad \text{for} \quad 0 \leq x < 1 \tag{3.11}$$

and

$$\bar{E}_{2q+1,2p}(x+1) = -\bar{E}_{2q+1,2p}(x) \qquad \text{for all } x, \tag{3.12}$$

we see that

$$\bar{E}_{2q+1,2p} \in \mathscr{P}^{2p}_{2q+1},$$

and we have the following theorem.

THEOREM 5. *Of all* $P(x) \in \mathscr{P}^{2p}_{2q+1}$, *only* $\bar{E}_{2q+1,2p}(x)$ *has least sup-norm.*

As corollaries to these two theorems, we have the following.

COROLLARY 3. *The polynomial* $E(x) = E_{2q,2p-1}(x)$ *is the unique polynomial satisfying the following conditions*:

(1) $E(x) = x^{2q}$ + *lower degree terms*;

(2) $E^{(\nu)}(0) = -E^{(\nu)}(1)$, $\nu = 0, 1,\ldots, 2p-1$;

(3) $E(x)$ *has least sup-norm in* $[0, 1]$.

COROLLARY 4. *The polynomial* $E(x) = E_{2q+1,2p}(x)$ *is the unique polynomial satisfying the following conditions*:

(1) $E(x) = x^{2q+1}$ + *lower degree terms*;

(2) $E^{(\nu)}(0) = -E^{(\nu)}(1)$, $\nu = 0, 1,\ldots, 2p$;

(3) $E(x)$ *has least sup-norm in* $[0, 1]$.

Curiously enough, the same perfect splines which settle the problem in $\mathscr{P}_m^r$ when m and r are of different parity also yield the desired result when the parity of m and r is the same.

THEOREM 6. *Of all* $P(x) \in \mathscr{P}^{2p-1}_{2q+1}$, $\bar{E}_{2q+1,2p}(x)$ *has least sup-norm.*

THEOREM 7. *Of all* $P(x) \in \mathscr{P}_{2q}^{2p-2}$, $\bar{E}_{2q,2p-1}(x)$ *has least sup-norm.*

The question of unicity here is as yet unsettled.

For low values of r, $r = 1$ and 2, the ET-polynomials can be easily constructed explicitly. For example, consider $E_{2q,1}$, and let α denote the least zero and β the largest zero of the Chebyshev polynomial $T_{2q}(x)$. Then we have the following theorem.

THEOREM 8. $E_{2q,1}(x) = 2^{-2q+1}(\beta - \alpha)^{-2q} T_{2q}(\alpha(1 - x) + \beta x)$.

This theorem is easily verified. In the first place, the right side of the above equality is even about $x = \frac{1}{2}$ and it is a polynomial with highest term x^{2q}. Also on $[0, \frac{1}{2}]$ the polynomial has q points of equioscillation, and this fact together with Corollary 1 establishes the theorem.

In a similar manner we can construct $E_{2q+1,2}(x)$. For starting with $T_{2q+1}(x)$ and denoting the least zero and the largest zero of $T'_{2q+1}(x)$ by α' and β' respectively, we can establish as above the following theorem.

THEOREM 9. $E_{2q+1,2}(x) = 2^{-2q}(\beta' - \alpha')^{-2q+1} T_{2q+1}(\alpha'(1 - x) + \beta' x)$.

4. TWO LEMMAS OF SCHOENBERG AND ZIEGLER CONCERNING ZEROS OF CARDINAL SPLINE FUNCTIONS

To state these two fundamental lemmas, we need some notation. Let $\mathscr{S}_n{}^r$ be the class of cardinal spline functions of degree n belonging to the continuity class $C^r(R)$ $(1 \leqslant r \leqslant n - 1)$. We denote by $\tilde{\mathscr{S}}_n{}^r$ the subclass of $\mathscr{S}_n{}^r$ consisting of all splines $S(x)$ such that

$$S(x) \not\equiv 0 \quad \text{in} \quad (n, n + 1) \qquad \text{for any } n. \tag{4.1}$$

We then count the zeros of any $S(x) \in \tilde{\mathscr{S}}_n{}^r$ in the following manner: If z is not a knot, we have a zero of multiplicity k provided

$$S(z) = S'(z) = \cdots = S^{(k-1)}(z) = 0,$$

while

$$S^{(k)}(z) \neq 0.$$

If z is a knot of multiplicity l, we may use the same definition for $k \leqslant n - l$.

Now let $Z\{S(x); [a, b]\}$ denote the number of zeros of $S(x)$, counting multiplicities as above, in $[a, b]$.

LEMMA 1. *Let* $S(x) \in \tilde{\mathscr{S}}_n{}^r$. *Then*

$$Z\{S(x); [0, k]\} \leqslant n + (k - 1)(n - r). \tag{4.2}$$

The second lemma concerns functions which weakly oscillate about zero and yields a lower bound on the number of zeros of such a function.

LEMMA 2. *Let $S(x) \in \tilde{\mathcal{S}}_n^r$, $r \geq 1$, and assume that there exist points*

$$0 \leq x_1 < x_2 < \cdots < x_{2s} < 1, \tag{4.3}$$

such that with $\epsilon = +1$ *or* -1,

$$\epsilon(-1)^i S(x_i + n) \geq 0, \qquad i = 1, 2,..., 2s; \qquad n = 0, 1,... . \tag{4.4}$$

Then

$$Z\{S(x); [0, k]\} \geq 2sk - 1, \qquad k = 1, 2,... . \tag{4.5}$$

For the proofs of these two lemmas, we refer the reader to the paper of Schoenberg and Ziegler [9].

5. PROOFS OF THEOREMS 2 AND 3

Proof of Theorem 2. Suppose, to the contrary, that the collection $\tilde{E}_{2p}(x), \tilde{E}_{2p+2}(x),..., \tilde{E}_{2q}(x)$ does not form a Chebyshev set. Then there exist $q - p + 1$ points $0 < x_1 < x_2 < \cdots < x_{q-p+1} \leq \frac{1}{2}$ which are zeros for the nontrivial polynomial

$$Q(x) = c_1\tilde{E}_{2p}(x) + \cdots + c_{q-p+1}\tilde{E}_{2q}(x) \qquad (0 \leq x \leq \tfrac{1}{2}).$$

Then by (3.2)

$$xQ(x) = c_1E_{2p}(x) + \cdots + c_{q-p+1}E_{2q}(x) \qquad (0 \leq x \leq \tfrac{1}{2})$$

and $xQ(x)$ must have zeros at $0, x_1, x_2,..., x_{q-p+1}$.

Now consider the extension of $xQ(x)$ defined by

$$S(x) = \overline{xQ(x)} = c_1\overline{E}_{2p}(x) + \cdots + c_{q-p+1}\overline{E}_{2q}(x).$$

Then clearly $S(x) \in \mathcal{S}_{2q}^{2p-1}$, and in each interval $[\nu, \nu + 1)$ $S(x)$ has at least $2q - 2p + 2$ zeros. This is true even if $x_{q-p+1} = \frac{1}{2}$, for then $x = \frac{1}{2}$ must be a double zero because of the evenness of E_{2j}.

Thus, we clearly must have

$$Z\{S(x); [0, k]\} \geq k(2q - 2p + 2).$$

On the other hand from Lemma 1 we obtain

$$Z\{S(x); [0, k]\} \leq 2q + (k - 1)(2q - 2p + 1).$$

But for sufficiently large k these two inequalities are contradictory, and so our original system must in fact be a Chebyshev system. ∎

Proof of Theorem 3. Suppose, to the contrary, that the collection $\tilde{E}_{2p+1}(x), \tilde{E}_{2p+3}(x), \ldots, \tilde{E}_{2q+1}(x)$ does not form a Chebyshev set. Then there exist $q-p+1$ points $0 \leq x_1 < x_2 < \cdots < x_{q-p+1} \leq \frac{1}{2}$ which are zeros for the nontrivial polynomial

$$Q(x) = c_1\tilde{E}_{2p+1}(x) + c_2\tilde{E}_{2p+3}(x) + \cdots + c_{q-p+1}\tilde{E}_{2q+1}(x).$$

Then by (3.1)

$$(x - \tfrac{1}{2})\, Q(x) = c_1 E_{2p+1}(x) + c_2 E_{2p+3}(x) + \cdots + c_{q-p+1} E_{2q+1}(x)$$

and $(x - \frac{1}{2})\, Q(x)$ must have zeros at

$$x_1 < x_2 < \cdots < x_{q-p+1} \leq x_{q-p+2} = \tfrac{1}{2}.$$

Now consider the extension of $(x - \frac{1}{2})\, Q(x)$ defined by

$$\begin{aligned} S(x) &= \overline{(x - \tfrac{1}{2})\, Q(x)} \\ &= c_1\bar{E}_{2p+1}(x) + c_2\bar{E}_{2p+3}(x) + \cdots + c_{q-p+1}\bar{E}_{2q+1}(x). \end{aligned}$$

Then clearly $S(x) \in \mathscr{S}^{2p}_{2q+1}$. Counting the zeros of $S(x)$ in any interval $[\nu, \nu+1)$, we find:

if $x_1 > 0$, $x_{q-p+1} < \frac{1}{2}$, then $2q - 2p + 3$ zeros;
if $x_1 = 0$, $x_{q-p+1} < \frac{1}{2}$, then $2q - 2p + 2$ zeros;
if $x_1 > 0$, $x_{q-p+1} = \frac{1}{2}$, then $2q - 2p + 3$ zeros;
if $x_1 = 0$, $x_{q-p+1} = \frac{1}{2}$, then $2q - 2p + 2$ zeros.

So in any case we have

$$Z\{S(x); [0, k]\} \geq k(2q - 2p + 2),$$

while by Lemma 1

$$Z\{S(x); [0, k]\} \leq 2q + 1 + (k - 1)(2q - 2p + 1).$$

Again for sufficiently large k these two inequalities are contradictory, and so our set must be a Chebyshev system. ∎

6. Proof of Proposition 1

Clearly from the linear theory, we know that there exists a polynomial of best approximation (see Lorentz [4, p. 17]). Suppose

$$P(x) = \sum_{i=1}^{k} c_i g_i(x) \tag{6.1}$$

is such a best approximation to $F(x)$, i.e.,

$$\| F(x) - P(x) \| = \zeta \leq \| F(x) - \tilde{P}(x) \|,$$

where $\tilde{P}(x)$ is any other linear combination of the form (6.1). We show that

$$h(x) = F(x) - P(x)$$

must take on the values $\pm\zeta$ with alternating signs at $k+1$ points.

Set

$$A^+ = \{x \in [a, b] \mid h(x) = \zeta\},$$
$$A^- = \{x \in [a, b] \mid h(x) = -\zeta\}$$

and

$$A = A^+ \cup A^-.$$

Clearly these sets are closed and $A^+ \cap A^- = \emptyset$. Now if there does not exist a $k+1$ point equioscillation, then we must be able to divide $[a, b]$ into at most k mutually disjoint open intervals

$$I_1 < I_2 < \cdots < I_j \qquad j \leq k$$

such that

$$A^+ \subset \bigcup I_{2i-1}, \qquad A^- \subset \bigcup I_{2i} \tag{6.2}$$

(or perhaps the other way around).

Now using the Chebyshev property of the $f_i(x)$, we can easily construct

$$Q(x) = \sum_{i=1}^{k} b_i f_i(x)$$

so that

$$Q(x) < 0 \quad \text{on} \quad \bigcup I_{2i-1}, \qquad Q(x) > 0 \quad \text{on} \quad \bigcup I_{2i}. \tag{6.3}$$

It follows that

$$\begin{aligned} (x-a)\,Q(x) &< 0 \qquad \text{on} \quad \bigcup I_{2i-1}, \\ (x-a)\,Q(x) &> 0 \qquad \text{on} \quad \bigcup I_{2i} \end{aligned} \tag{6.4}$$

and (6.2) and (6.4) together imply

$$\max_{x \in A} (x-a)\,Q(x)\,h(x) < 0. \tag{6.5}$$

(6.5) contradicts the Kolmogorov condition for a best approximation (see [4, p. 18]). So we conclude that $h(x)$ must exhibit $k+1$ points of equioscillation, as claimed.

Conversely, let us assume that there exists a $P(x)$ of the form (6.1) such that

$$h(x) = F(x) - P(x)$$

has $k+1$ points of equioscillation, and $\| h(x) \| = \zeta$. We then claim that $P(x)$ must be the unique best approximation to $F(x)$. For suppose there were to exist some $\tilde{P}(x)$ of the form (6.1) with

$$\| F(x) - \tilde{P}(x) \| \leqslant \zeta \tag{6.6}$$

and consider

$$\begin{aligned} g(x) &= (F(x) - \tilde{P}(x)) - (F(x) - P(x)) \\ &= P(x) - \tilde{P}(x). \end{aligned} \tag{6.7}$$

Now $g(a) = 0$ and there are k more zeros because of (6.6) and the equioscillation requirement. So $g(x)$ has at least $k+1$ zeros. But by (6.7) and (3.3)

$$g(x) = \sum_{i=1}^{k} c_i g_i(x) = (x-a) \sum_{i=1}^{k} c_i f_i(x) = (x-a) f(x),$$

where $f(x) = \sum_{i=1}^{k} c_i f_i(x)$. Then $f(x)$ must have k zeros, which contradicts our assumption that the $f_i(x)$ form a Chebyshev system. ∎

7. Proofs of Theorems 4 and 5

Turning our attention to Theorem 4, we consider $E_{2q,2p-1}(x)$ and denote by

$$0 < x_1 < x_2 < \cdots < x_{q-p+1} = \tfrac{1}{2}$$

its $q-p+1$ points of equioscillation. That $x_{q-p-1} = \frac{1}{2}$ is indeed a point of equioscillation for $E_{2q,2p-1}(x)$ is easily shown by arguments similar to those which established Theorems 2 and 3. Define

$$x_{2q-2p+2-i} = 1 - x_i \qquad i = 1, \ldots, q-p.$$

Then since $\bar{E}_{2q,2p-1}$ is even about $x = \frac{1}{2}$ and periodic, we see that the function attains extreme values with alternating signs at the $2q - 2p + 1$ points x_i in $(0, 1)$ and at all points congruent to these modulo 1.

Now to show that $E_{2q,2p-1}$ is of least sup-norm within the class $\mathscr{P}_{2q}^{2p-1}$, suppose to the contrary that there did exist $F(x) \in \mathscr{P}_{2q}^{2p-1}$ such that

$$\| F(x) \|_\infty < \| E_{2q,2p-1}(x) \|_\infty . \tag{7.1}$$

Consider the spline function

$$S(x) = \bar{E}_{2q,2p-1}(x) - F(x), \tag{7.2}$$

which we suppose does not vanish identically. Note that

$$S(x) \in \mathscr{S}_{2q-1}^{2p-1} . \tag{7.3}$$

In fact

$$S(x) \in \tilde{\mathscr{S}}_{2q-1}^{2p-1} \tag{7.4}$$

For if not, then we can assume that for some n

$$\begin{aligned} S(x) &\equiv 0 \qquad \text{if} \quad n - 1 \leq x \leq n, \\ S(x) &\not\equiv 0 \qquad \text{if} \quad n \leq x \leq n + 1. \end{aligned}$$

However, $S(x) \in C^{2p-1}(R)$, and so we must have

$$S(x) = \sum_{2p}^{2q-1} c_\nu (x - n)^\nu \qquad \text{for} \quad n \leq x \leq n + 1, \tag{7.5}$$

where not all of the c_ν vanish.

But by (7.1) and (7.2) together with the $2q - 2p + 1$ point equioscillation, we conclude that $S(x)$ must have $2q - 2p$ zeros in the interior of $[n, n + 1]$, and when we count the zero of multiplicity $2p - 1$ at $x = n$, we find that $S(x)$ has $2q$ zeros in $[n, n + 1]$, which is a contradiction since $S(x)$ is a polynomial of degree $2q - 1$ there. So $S(x) \in \tilde{\mathscr{S}}_{2q-1}^{2p-1}$.

So we can use Lemma 2 to conclude that

$$Z\{S(x); [0, k]\} \geq k(2q - 2p + 1) - 1. \tag{7.6}$$

On the other hand, Lemma 1 shows that

$$Z\{S(x); [0, k]\} \leq 2q - 1 + (k - 1)(2q - 2p)$$

which contradicts (7.6) for large k. This settles Theorem 4. ∎

Theorem 5 is handled in a similar manner, and so may be passed over without due concern.

8. Proofs of Theorems 6 and 7

For these two final theorems, we need a new approach. We will use the Rivlin–Shapiro criterion for a best approximation, but this criterion, like the Kolmogorov criterion, is only applicable when the space in which the approximation is sought is finite dimensional. It is well known that the space of spline functions with infinitely many knots is not finite dimensional, and, therefore, we will have to seek some way to transform our problem to a finite dimensional setting.

The way home is indicated by Theorems 4 and 5. There it turned out that the optimal splines satisfied the relation $F(x+1) = -F(x)$ for all x, and that this is no accident is shown by the following lemma.

LEMMA 3. *If $F(x)$ is an element of $\mathscr{P}_m{}^r$ of finite sup-norm ζ, then there exists an element $F^*(x)$ of $\mathscr{P}_m{}^r$ whose norm is $\leqslant \zeta$, and this element has the following periodicity relation*:

$$F^*(x+1) = -F^*(x) \qquad \textit{for all} \quad x.$$

Proof. Consider the sequence of functions

$$F_n(x) = 1/n \sum_{\nu=0}^{n-1} (-1)^\nu F(x+\nu) \qquad n = 1, 2,\ldots .$$

These functions obviously belong to $\mathscr{P}_m{}^r$ and moreover

$$\| F_n(x)\|_\infty \leqslant \zeta.$$

The sequence $\{F_n\}$ is compact. In fact, on the interval [0, 1], we have merely a uniformly bounded sequence of mth degree polynomials which, therefore, clearly has a convergent subsequence. This observation is valid for every subsequence in each unit interval, and so we can use the standard diagonal process to establish the existence of a subsequence $F_{n_i}(x)$ convergent on $(-\infty, \infty)$ and the convergence will be uniform on compact intervals.

Denote the limit function by $F^*(x)$. Then clearly $\| F^* \|_\infty \leqslant \zeta$, and $F^* \in \mathscr{P}_m{}^r$. As for the periodicity relation, consider

$$\begin{aligned} F_n(x+1) &= 1/n \sum_{\nu=0}^{n-1} (-1)^\nu F(x+1+\nu) \\ &= -1/n \sum_{\nu=0}^{n-1} (-1)^\nu F(x+\nu) + (1/n)(F(x) + (-1)^{n-1} F(x+n)). \end{aligned}$$

Letting $n = n_i \to \infty$, and using the boundedness of F on $(-\infty, \infty)$ we obtain

$$F^*(x+1) = -F^*(x). \quad \blacksquare$$

Before turning to the proof of Theorem 6, let us recall the Rivlin–Shapiro criterion for the real case ([6]; see also [4, Chapter 2, Section 3]). We consider a real Banach space $C[I]$, where I is any compact set, in particular [0, 1]. Also suppose we have a finite set of real-valued functions $\Phi \subset C[I]$, and let Q denote any element in the n-dimensional linear span of Φ. Then a (real) *signature* σ on I is a function whose support consists of a finite number of points and whose values are ± 1. Such a signature σ with support $S = \{x_1, ..., x_r\}$ will be called an *extremal signature* (with respect to the system Φ) if there exists a function μ with support S for which

$$\operatorname{sign} \mu(x_k) = \sigma(x_k) \qquad k = 1, 2, ..., r$$

and

$$\sum_{k=1}^{r} \mu(x_k)\, Q(x_k) = 0$$

for all Φ-polynomials Q. Then we can state the following.

CRITERION (Rivlin–Shapiro). *A Φ-polynomial P, not identically equal to the function $f \in C[I]$, is a polynomial of best approximation for f if and only if there exists an extremal signature σ with support $S = \{x_1, ..., x_r\}$ contained in the set of equioscillations of $f - P$ and such that*

$$r \leqslant n + 1$$

and

$$\operatorname{sign}[f(x_k) - P(x_k)] = \sigma(x_k) \qquad k = 1, 2, ..., r.$$

We can now turn to Theorem 6. From Corollary 4 we have that $E_{2q+1,2p}(x)$ is of all polynomials of the form

$$E_{2q+1}(x) + \sum_{\nu=2p+1}^{2q} c_\nu E_\nu(x)$$

the one which has least sup-norm on [0, 1]. This may also be viewed as the error function obtained when approximating $E_{2q+1}(x)$ by linear combinations of the Euler polynomials $E_\nu(x)$, $\nu = 2p+1, ..., 2q$, in total $2q - 2p$ functions. We would expect a $2q - 2p + 1$ point equioscillation property for the error function, but in fact on [0, 1], $E_{2q+1,2p}$ has $2q - 2p + 2$ points of equioscillation. This difference leads us to suspect that even if we increase the dimension of the approximating space by adjoining the function $E_{2p}(x)$ we will still get the same best approximation to $E_{2q+1}(x)$. And in fact the above Lemma 3 shows that this observation is essentially the content of Theorem 6.

Proof of Theorem 6. Using the Rivlin–Shapiro criterion, we show that $E_{2q+1,2p}(x)$ is a best approximation of $E_{2q+1}(x)$ by elements of the $2q-2p+1$ dimensional subspace

$$E_{2p}(x), E_{2p+1}(x), \ldots, E_{2q}(x). \tag{8.1}$$

As before let x_i', $i = 1, \ldots, q-p+1$, denote the equioscillations of $E_{2q+1,2p}(x)$ in $[0, \frac{1}{2})$ described in Corollary 2. Define

$$x'_{2q-2p+3-i} = 1 - x_i' \qquad i = 1, \ldots, q-p+1.$$

The set $\{x_i' \mid i = 1, \ldots, 2q-2p+2\}$ exhibits all the points of equioscillation in $[0, 1]$ for $E_{2q+1,2p}(x)$. To show that the Rivlin–Shapiro criterion is satisfied, we must prove that there exists a set of weights $w_1, \ldots, w_{2q-2p+2}$ such that

$$\sum_{j=1}^{2q-2p+2} w_j E_\nu(x_j') = 0 \qquad \nu = 2p, 2p+1, \ldots, 2q, \tag{8.2}$$

and

$$\text{the weights alternate in sign.} \tag{8.3}$$

Select skew-symmetric weights, as follows:

$$\begin{aligned} w_i &= (-1)^{i+1} P_i, \\ w_{2q-2p+3-i} &= -w_i, \end{aligned} \qquad i = 1, 2, \ldots, q-p+1, \tag{8.4}$$

where the P_i's are positive and are to be determined.

By the symmetries involved, relation (8.2) is valid for all even degree polynomials $E_{2\nu}(x)$. We need only prove that for a suitable choice of positive numbers $P_1, \ldots, P_{q-p+1}$ (8.2) will hold for the odd degree Euler polynomials. The odd symmetry of $E_{2\nu+1}(x)$ about $\frac{1}{2}$ indicates that (8.2) will hold if the P_i's satisfy

$$\sum_{j=1}^{q-p+1} (-1)^{j+1} P_j E_{2\nu+1}(x_j') = 0, \qquad \nu = p, p+1, \ldots, q-1. \tag{8.5}$$

This amounts to a homogeneous system of $q-p$ equations in the $q-p+1$ unknowns P_j. It follows easily from Theorem 3 that all minors of order $q-p$ of the coefficient matrix of this system are strictly of one sign, and, therefore, (8.5) admits a solution with $P_j > 0$. So the Rivlin–Shapiro criterion is satisfied, and Theorem 6 is complete. ∎

Proof of Theorem 7. Here we proceed much like above. We wish to prove that $E_{2q,2p-1}(x)$ is the best approximation to $E_{2q}(x)$ using linear combinations of the $2q - 2p + 1$ functions

$$E_{2p-1}(x), E_{2p}(x), \ldots, E_{2q-1}(x).$$

Denote as in Corollary 1 the $q - p + 1$ points of equioscillation of $E_{2q,p-1}(x)$

$$0 < x_1 < x_2 < \cdots < x_{q-p+1} = \tfrac{1}{2}.$$

and define

$$x_{2q-2p+2-i} = 1 - x_i \qquad i = 1, \ldots, q-p.$$

Then the set $\{x_i : i = 1, 2, \ldots, 2q - 2p + 1\}$ exhibits all the points of equioscillation in $[0, 1]$ of $E_{2q,2p-1}(x)$. We now must exhibit a set of weights $w_1, w_2, \ldots, w_{2q-2p+1}$ such that

$$\sum_{j=1}^{2q-2p+1} w_j E_\nu(x_j) = 0 \qquad \nu = 2p-1, \ldots, 2q-1 \tag{8.6}$$

and

$$\text{the weights alternate in sign.} \tag{8.7}$$

We select symmetric weights as follows:

$$\begin{aligned} w_i &= (-1)^{i+1} P_i, \qquad && i = 1, 2, \ldots, q-p+1, \\ w_{2q-2p+2-i} &= w_i, && i = 1, 2, \ldots, q-p, \end{aligned} \tag{8.8}$$

where $P_i > 0$. These weights clearly satisfy (8.7) and due to the symmetries involved (8.6) is valid for the odd degree Euler polynomials. For the even degrees, (8.6) is a consequence of

$$\sum_{j=1}^{q-p+1} (-1)^{j+1} P_j E_{2\nu}(x_j) = 0, \qquad \nu = p, p+1, \ldots, q-1, \tag{8.9}$$

a system of $q - p$ equations in $q - p + 1$ unknowns, and as before by Theorem 2 this system has a positive solution $P_j > 0$.

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